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SIMPLE EXAMPLES OF COMPLEX PHENOMENA IN PLASTICITY

by

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ABSTRACT

In 1951 D. C. Drucker presented a classic example of a simple three-bar system in which a single monotonically-increasing external load produced an actual plastic stress reversal with one bar yielding first in tension and then in compression. This example is reviewed and built upon. Other simple examples illustrate general loading, unloading, and reloading of elastic/plastic structures, shakedown, and lack of uniqueness in contained plastic deformation.

1. INTRODUCTION

There are many complications in applications of plasticity theory, particularly in comparison with the linear theory of elasticity. If these complications are encountered for the first time in a "real" problem with large amounts of analysis and computation even for an elastic material, the peculiar features of plastic behavior may be masked, and the probability of undetected errors is greatly increased. For this reason, there is an obvious advantage to discussing plasticity first in terms of simple structures, the simpler the better. Of course, if a problem is too simple, then some essential features of plasticity may not appear. However, one of the surprising features of plasticity theory is that so many of its important features can be illustrated with problems which are easily comprehended by students in a beginning class in statics.

In the present paper we shall illustrate several of the features of plasticity theory with two very simple three-bar statistically indeterminate trusses, as shown in Figs. 1 and 8. Specifically, in Sec. 2 we will use the truss of Fig. 1 to show the difference between hardening and perfectly-plastic materials, and the effect of including or neglecting elastic strain components. Section 3 extends these results to unloading and reloading. We then examine some undesirable effects which may occur when loads are applied repeatedly. The next two sections illustrate types of behavior which are drastically different from those encountered in elasticity. In Sec. 5 we look at the truss in Fig. 8 and show that under a monotonically increasing single load a bar may yield first in tension and then in compression. Thus proportional "loading" does not by any means guarantee proportional "stressing". Section 6 returns to the truss in Fig. 1 and shows that a boundary-value problem that is "well-posed" in the theory of elasticity may have a non-unique solution in plasticity. The final section of the paper will quote analytic and numerical solutions to more complex problems where the *same types of responses* are observed.

2. LOADING OF DIFFERENT PLASTICITY MODELS

The defining equations for any problem in continuum or structural mechanics come from three sources: statics, kinematics, and *constitutive*. For the truss in Fig. 1, the two statics equations are obtained from horizontal and vertical equilibrium of point D:

$$(F_1 - F_3)/\sqrt{2} = H \qquad (F_1 + F_3)/\sqrt{2} + F_2 = V \tag{1}$$

where F_i is the tensile force in bar i. The kinematics expresses the elongation e_i of each bar in terms of the displacements u and v of point D:

$$e_1 = (u + v)/\sqrt{2} \qquad e_2 = v \qquad e_3 = (-u + v)/\sqrt{2} \tag{2}$$

Obviously, we are using a strictly linear theory in writing Eqs. (1) and (2).

In the present section we are considering only the case where stress and strain are both increasing functions of time, so that we can quite generally write

$$\sigma = f(\varepsilon) \tag{3}$$

Therefore, for each bar of the truss we may write

$$F_i = A_i f_i(\varepsilon_i)/L_i \tag{4}$$

where A_i is the cross-sectional area and L_i is the length of bar i. Although $f(\varepsilon)$ could represent an experimentally-obtained stress-strain curve, we will consider here the four piecewise-linear curves model shown in Fig. 2. *Figure 2a* portrays an "elastic/strain-hardening" material (E/SH) whose equations may be written

$$\begin{aligned} 0 < \sigma < \sigma_y \qquad & \sigma = E\varepsilon \\ \sigma_y < \sigma \qquad & \sigma = E'\varepsilon + (1 - E'/E)\sigma_y \end{aligned} \tag{5}$$

where E is the elastic modulus, E' is the plastic modulus, and σ_y is the yield stress. We define the yield force, elastic stiffness, and plastic stiffness of bar i by

$$Y_i = A_i\sigma_{yi} \qquad k_i = A_iE_i/L_i \qquad k_i' = A_iE_i'/L_i \tag{6}$$

respectively, to obtain the constitutive equation *for* bar *i*:

$$\text{E/} \qquad 0 < F_i < Y_i \qquad F_i = k_ie_i \tag{7a}$$

$$\text{E/SH} \qquad Y_i < F_i \qquad F_i = k_i'e_i + Y_i(1 - k_i'/k_i) \tag{7b}$$

Similarly, the constitutive equation *for the* "elastic/perfectly-plastic" material (E/PP) in Fig. 2b consists of Eq. (7a) for $0 < F_i < Y_i$ and

$$\text{E/PP} \qquad e_i > Y_i/k_i \qquad F_i = Y_i \tag{7c}$$

For the "rigid/strain-hardening" (R/SH) and "rigid-perfectly-plastic" (R/PP) materials in Figs. 2c and 2d, respectively, the equations are

$$\text{R/} \qquad F_i < Y_i \qquad e_i = 0 \tag{7d}$$

$$\text{R/SH} \qquad F_i > Y_i \qquad F_i = k_i'e_i + Y_i \tag{7e}$$

$$\text{R/PP} \qquad e_i > 0 \qquad F_i = Y_i \tag{7f}$$

Let the three bars of the truss in Fig. 1 all have the same moduli E and E', the same area A, and the same yield stress σ_Y; the yield force in each bar will then have the same value Y. We define $k = AE/L$. Then the bar stiffnesses are

$$k_1 = k_3 = k/\sqrt{2} \qquad k_2 = k \tag{8}$$

We consider a deformation-controlled loading program in which $u = 0$ and v is slowly increased. It then follows from Eqs. (2) that

$$e_1 = e_3 = v/\sqrt{2} \qquad e_2 = v \tag{9}$$

Regardless of which of Eqs. (7) apply, we see that $F_1 = F_3$, whence Eqs. (1) can be written

$$H = 0 \qquad V = \sqrt{2}F_1 + F_2 \tag{10}$$

We first examine the E/SH material. For v sufficiently small all bars will be elastic so that (7a) applies. Thus

Stage 1, all elastic $\quad F_1 = F_3 = kv/2 \qquad F_2 = kv \qquad$ (11a,b)

$$V = kv(1 + 1/\sqrt{2}) \tag{11c}$$

This stage will reach its limit when bar 2 yields at $kv = Y$. At this point we have

Stage 1L $\quad kv/Y = 1 \qquad F_1 = F_3 = Y/2 \qquad F_2 = Y \qquad$ (12)

$$V = (1 + 1/\sqrt{2})Y$$

In the next stage bar 2 is plastic and is governed by Eq. (7b) but (7a) still applies to bars 1 and 3. Thus Eqs. (11) are replaced by

Stage 2, bar 2 plastic $\quad$ (13)

$$F_1 = F_3 = kv/2$$
$$F_2 = k'v + Y(1 - k'/k)$$
$$V = kv(1/\sqrt{2} + k'/k) + Y(1 - k'/k)$$

When $kv = 2Y$ bars 1 and 3 reach yield with the result

Stage 2L $\quad$ (14)

$$kv/Y = 2 \qquad F_1 = F_3 = Y \qquad F_2 = Y(1 + k'/k)$$
$$V = Y(\sqrt{2} + 1 + k'/k)$$

As v is still further increased, all bars will be plastic and the solution from now on will be

Stage 3, all plastic $\quad$ (15)

$$F_1 = F_3 = k'v/2 + Y(1 - k'/k)$$
$$F_2 = k'v + Y(1 - k'/k)$$
$$V = k'v(1/\sqrt{2} + 1) + (\sqrt{2} + 1)Y(1 - k'/k)$$

The solid curve in Fig. 3 shows the resulting load-deflection curve defined by Eqs. (11-15).

For the E/PP model of Fig. 1b, the analysis is even simpler. Stage 1 and Stage 1L are, of course, still given by Eqs. (11) and (12), respectively. In Stage 2 bar 2 has the constant value of Y and bars 1 and 3 are elastic. Thus (11a) and (1) lead immediately to

Stage 2, bar 2 plastic $\quad$ (16)

$$F_1 = F_3 = kv/2 \qquad F_2 = Y$$
$$V = kv/\sqrt{2} + Y$$

Equations (16) may also be obtained from Eqs. (13) by setting $k' = 0$.

Stage 2L again occurs when bars 1 and 3 reach yield with the result

Stage 2L, all plastic $\quad F_1 = F_3 = F_2 = Y \qquad V = (\sqrt{2} + 1)Y \qquad$ (17a)

$$kv/Y = 2 \tag{17b}$$

At this point the perfectly-plastic truss cannot tolerate any further increase in load, but can continue to deform indefinitely with no further load increase. Thus the solution for Stage 3 consists of Eqs. (17a) with any value of v greater than 2Y/k. The dashed curve in Fig. 3 shows the load-deformation curve for an E/PP material as defined by Eqs. (11, 12, 16, 17).

If the truss is made of a R/SH material, it will not be able to deform at all until all three bars have reached yield. Thus Stages 1 and 2 of the previous models are condensed into a single Stage 1 with · · The bar forces are non-unique until Stage IL is reached when they are given by Eqs. (17a) with v = 0. For any positive v, then, all three bars are plastic, and the solution is

$$\text{Stage 2, all plastic} \quad F_1 = F_3 = Y + k'v/2 \qquad F_2 = Y + k'v$$
$$V = k'v(1/\sqrt{2} + 1) + (\sqrt{2} + 1)Y \tag{18}$$

The dot-dashed curve in Fig. 3 shows the resulting load-deflection curve.

Finally, the R/PP model predicts zero displacement until Stage 1L as given by Eqs. (17a) with v = 0. For Stage 2 the static quantities are still given by (17a) with any positive v. The load-deflection curve is shown dotted in Fig. 3.

Figure 3 is drawn with dimensionless scales with an assumed ratio k/k' = 0.1. Let us define the "yield-point load" of the truss as that value $V = V_0$ for which the truss becomes very much more flexible. For the E/SH model it is given by the last Eq. (14), and for the other three models it has the precise value $Y(\sqrt{2} + 1)$. Therefore, if only V_0 is required, even the simplest R/PP model gives a very good approximation to it.

For values of the load less than the yield-point load, the effect of strain hardening is very small, and it appears quite reasonable to use the E/PP model as an approximation of the more accurate E/SH one. On the other hand, if the applied load is to be much larger than V_0, then the effect of elastic strains becomes increasingly less important, and the R/SH model gives a good approximation.

3. UNLOADING AND RELOADING

Equations (5) are no longer adequate to describe the constitutive behavior if the stress in a bar is allowed to both increase and decrease. Instead, we must formulate them in terms of rates or increments. To this end, at any given instant we characterize the behavior of a bar as either elastic (E), strain-hardening (SH), or perfectly-plastic (PP). Then the rate equations corresponding to (5) may be written

$$\text{E:} \quad \dot{\sigma} = E\dot{\epsilon} \tag{19a}$$

$$\text{SH:} \quad \dot{\sigma} = E'\dot{\epsilon} \tag{19b}$$

$$\text{PP:} \quad \dot{\sigma} = 0 \tag{19c}$$

Equations (19) do not form a complete description of material behavior, because we still require rules to tell which set of equations are applicable at any given time. For the PP material this description is quite simple. The magnitude of the stress can never exceed the initial yield stress σ_y. If $\sigma = \sigma_y$ and ϵ is increasing or if $\sigma = -\sigma_y$ and ϵ is decreasing, then the bar is plastic; otherwise it is elastic.

For the SH material, the yield stress does not remain constant. Rather, at any time there exist two numbers σ' and σ'' which represent the current yield stresses. If these are known, we can write the syllogism

$$\text{IF } (\sigma = \sigma' \text{ AND } \dot{\varepsilon} > 0) \text{ OR } (\sigma = \sigma'' \text{ AND } \dot{\varepsilon} < 0) \quad \text{THEN (PLASTIC) ELSE (ELASTIC)} \tag{19d}$$

Equation (19d) is supplemented with requirement

$$\sigma'' \leq \sigma \leq \sigma' \tag{19e}$$

Equations (19d,e) apply to a PP material if we assign the constant values

$$\text{PP:} \quad \sigma' = \sigma_y \qquad \sigma'' = -\sigma_y \tag{19f}$$

For the SH material it is still necessary to formulate a rule for the variation of the current yield stresses. We begin by stating that whenever a bar is elastic its yield stresses do not change. Next, we require that during plastic behavior the "active" yield stress remains equal to the bar stress:

$$\begin{aligned} \text{SH:} \quad & \text{IF (PLASTIC TENSION) THEN } \sigma' = \sigma \\ & \text{IF (PLASTIC COMPRESSION) THEN } \sigma'' = \sigma \end{aligned} \tag{19g}$$

Various models have been suggested for describing the change in the "*passive*" yield stress, i.e., the change in σ'' when $\sigma = \sigma'$. We consider here two common models known as "isotropic" (IH) and "kinematic" (KH) hardening. For an IH material the two yield stresses are always equal in magnitude and opposite in sign, whereas for a KH material the "elastic range" between the two yield stresses maintains a constant value:

$$\text{IH:} \quad \sigma' + \sigma'' = 0 \tag{19h}$$

$$\text{KH:} \quad \sigma' - \sigma'' = 2\sigma_y \tag{19i}$$

Figure 4 shows typical load-unload-reload curves for the three different materials.

We consider again the truss in Fig. 1 with all three bars having equal areas, moduli, and yield forces. As a loading program we hold u = 0, always. The vertical displacement v is first increased to 8Y/k, then decreased to -6Y/k, and finally increased to 0. As in the previous section all branches of the constitutive equations lead to $F_1 = F_3$ and H = 0. Therefore, combining the rate form of the kinematic equations with Eqs. (19a-c) we obtain the constitutive equations in the form

$$\text{R:} \quad \dot{F}_1 = k\dot{v}/2 \qquad \dot{F}_2 = k\dot{v} \tag{20a}$$

$$\text{SH:} \quad \dot{F}_1 = k'\dot{v}/2 \qquad \dot{F}_2 = k'\dot{v} \tag{20b}$$

$$\text{PP:} \quad \dot{F}_1 = 0 \qquad \dot{F}_2 = 0 \tag{20c}$$

For the remainder of Eqs. (19) one merely needs to replace σ, ε, and σ_y by, respectively, F_i, e_i, and Y_i.

Since we are analyzing a displacement-controlled program, it is not necessary to write the equilibrium equations in rate form. Rather, we carry out the incremental analysis for forces and displacements, combine to obtain resultant values of the forces, and then find the required external force V from the second Eq. (10).

The solution through Stage 3 is the same as in Sec. 2, and Stage 3L is obtained by setting kv/Y = 8 in Stage 3:

$$\text{Stage 3L} \quad kv/Y = 8 \qquad F_1/Y = 1 + 3r \qquad F_2/Y = 1 + 7r \tag{21}$$

$$V/Y = (\sqrt{2} + 1) + r(3\sqrt{2} + 7) \tag{22}$$

where we have defined

$$r = k'/k \tag{23}$$

In stage 4 we decrease v which will cause all bars to revert elastic behavior until bar 2 yields in compression. For KH the total change in F_2 during stage 4 will be $\Delta F_2 = -2Y$. Therefore, using (20a) for all three bars we may write the total change in solution during stage 4 as

$$\Delta F_2/Y = \Delta vk/Y = -2 \qquad \Delta F_1/Y = -1 \tag{24}$$

Adding these increments to the values at Stage 3L, we obtain

$$\text{Stage 4L(KH):} \quad kv/Y = 6 \qquad F_1/Y = 3r \qquad F_2/Y = -1 + 7r \tag{25}$$

In Stage 5 bar 2 is plastic so we use (20a) for bar 1 and (20b) for bar 2. This stage will end when bar 1 becomes plastic, so the total increments are

$$\Delta F_1/Y = -1 \qquad \Delta vk/Y = -2 \qquad \Delta F_2/Y = -2r \tag{26}$$

Addition of Eqs. (25) and (26) then gives

$$\text{Stage 5L(KH):} \quad kv/Y = 4 \qquad F_1/y = -1 + 3r \qquad F_2/Y = -1 + 5r \tag{27}$$

For the rest of the unloading both bars are plastic and take Eq. (20b). This stage ends when we again reverse v and start to reload. During reloading we enter stages 7, 8, and 9 corresponding to all bars elastic, bar 2 plastic, and all bars plastic respectively. The various solutions are

$$\text{Stage 6L(KH):} \quad kv/Y = -6 \qquad F_1/Y = -1 - 2r \qquad F_2/Y = -1 - 5r \tag{28a}$$

$$\text{Stage 7L(KH):} \quad kv/Y = -4 \qquad F_1/Y = -2r \qquad F_2/Y = 1 - 5r \tag{28b}$$

$$\text{Stage 8L(KH):} \quad kv/Y = -2 \qquad F_1/Y = 1 - 2r \qquad F_2/Y = 1 - 3r \tag{28c}$$

$$\text{Stage 9L(KH):} \quad kv/Y = 0 \qquad F_1/Y = 1 - r \qquad F_2/Y = 1 - r \tag{28d}$$

The load in each stage is given by the second Eq. (10).

For an IH material the solution will be the same up through Stage 3L, and the truss will have the same rates in Stage 4. However, we must now keep track of the current yield force Y' for each bar. During Stage 2 Y_2' will increase and remain equal to F_2, and during Stage 3 both yield forces will increase with the bar forces. Thus at Stage 3L the yield forces will be

$$\text{Stage 3L(IH):} \quad Y_1' = 1 + 3r \qquad Y_2' = 1 + 7r \tag{29}$$

and these values will continue to hold in Stage 4. Stage 4L will occur when $F_1 = -Y_1'$, hence

$$\text{Stage 4L(IH):} \quad F_1/Y \quad -4r \qquad F_2/Y = -(1 + 7r) \qquad kv/Y = 6 - 14r \tag{30}$$

During Stage 5 Y_2' will increase with the magnitude of F_2, but Y_1' will still be given by the first Eq. (30). At the end of Stage 5 we have

$$\text{Stage 5L(IH):} \quad \begin{aligned} &F_1/Y = -(1 + 3r) \qquad kv/Y = 4 - 12r \\ &F_2/Y = -(1 + 9r - 2r^2) \end{aligned} \tag{31}$$

Continuing in this fashion we obtain the solutions for the rest of the prescribed program. The results are summarized in Table 1. If $r > 0.0674$ for an IH material, the yield force in bar 1 will increase to the point that v = 0 during reloading while bar 1 is still elastic.

The results for a PP material can be obtained from either of the above SH solutions by setting r = 0, or can easily be found directly. The results are summarized in Table 2.

For all solutions the external load V is found by substituting the listed bar forces in the second Eq. (10). *Figure 5 shows the* resulting load-displacement curves for the three different materials. Results for the rigid/plastic materials will, of course, be very similar.

4. SHAKEDOWN

We consider first an E/PP material which is subjected to a prescribed loading program which continues indefinitely, usually as a *repeated cycle*. *If the loads are always* sufficiently small, the structure will remain everywhere elastic, and is of no concern in the present context. At the other extreme, if the loads ever exceed the yield-point load of the structure, there will be no equilibrium solution. The structure will collapse catastrophically, and the rest of the loading program will be meaningless. In the present section we are concerned with loading programs between these two extremes, i.e., the loading program includes *loads which* exceed the elastic limit but which nowhere exceed the yield-point load.

We illustrate various loading programs with regard to the truss in Fig. 1. *In every case we take* all three bars to have the same cross-sectional area and the same material properties.

Consider first a program in which H = 0, always, and V oscillates between 0 and 2.2Y. During the initial increase of V Eqs. (11), (12), and (16) apply but stage 2 ends with the limiting value of V:

$$\text{Stage 2L:}\quad F_1 = F_3 = 0.6\sqrt{2}Y \quad F_2 = Y \quad kv/Y = 1.2\sqrt{2} \quad V = 2.2Y \tag{32}$$

When V is decreased from 2.2Y, the changes in both bars will be elastic. Therefore, it follows from (20a) and the second Eq. (10) that

$$\Delta F_1 = 0.5k\Delta v \quad \Delta F_2 = k\Delta v \quad \Delta V = (1 + 1/\sqrt{2})k\Delta v \tag{33}$$

Setting $\Delta V = -2.2Y$ to reduce V to zero, *leads to the values*

$$\text{Stage 3L:}\quad F_1 = F_3 = (1.7\sqrt{2} - 2.2)Y \quad F_2 = (2.2\sqrt{2} - 3.4)Y \qquad kv/Y = 3.4\sqrt{2} - 4.4 \tag{34}$$

Since both forces are within the elastic range, this solution is the valid one for Stage 3L. Further, when V is again increased to 2.2Y, Eqs. (33) will hold for the entire reloading process, and the solution at stage 4L will be exactly the same as that at Stage 2L as given by Eqs. (32). Clearly further cycles will simply alternate between Eqs. (32) and (34) with both bars continuing to behave elastically.

The above behavior can be summarized as follows. There is a limited amount of plastic flow at the beginning of the loading process, but after this has taken place the rest of the cycle is completely elastic. A process with this property is called a "shakedown" process.

As an example of a process which does not shake down, suppose that after the initial loading to 2.2Y the load V is alternated between -2.2Y and +2.2Y. Stages 1 and 2 will be the same as before, and the change in Stage 3 will still be given by Eqs. (33). However this solution is valid only until $\Delta F_2 = -2Y$, which leads to

Stage 3L: $F_1 = (0.6\sqrt{2} - 1)Y$ $F_2 = -Y$ $kv/Y = 1.2\sqrt{2} - 2$ $V = -(0.4 + \sqrt{2})Y$ (35)

Further unloading takes place with bar 2 plastic, hence $\Delta F_2 = 0$. Choosing ΔV to bring V to its final value of -2.2Y, using Eq. (20a) for ΔF_1, and the second (10) for V, we obtain

Stage 4L: $F_1 = -0.3\sqrt{2}Y$ $F_2 = -Y$ $kv/Y = -0.6\sqrt{2}$ $V = -2.2Y$ (36)

The reloading process is similar, consisting of Stage 5 with all bars elastic and Stage 6 with bar 2 plastic in tension. The results are summarized in Table 3a. We observe that the solution for Stage 6L is exactly the same as the one at Stage 2L, so that this cycle of 4 stages will be exactly repeated in each cycle of V from +2.2Y to -2.2Y and back to +2.2Y as given in Table 3a. In particular, the value of the displacement will never exceed 1.697Y/k which is less than double the maximum elastic displacement.

At first glance it might appear that this loading program was a safe one for the truss since bar 1 never yields and the displacement is always bounded. However, from a materials point of view, the process of plastic displacement is not a reversible one. Although the gross displacement of bar 2 is the same after a complete cycle, the tensile plastic behavior in Stage 6 does not "undo" the compressive plastic elongation in Stage 4, but rather superimposes a tensile plastic elongation on it. To obtain some insight as to why this behavior is undesirable, we introduce the concept of "plastic work".

The differential internal work done on a bar is given quite generally by $dW = F\,de$. For an E/PP bar at yield there is no change in the bar force and hence no change in the elastic

elongation, so that all work done is plastic. Further, since the force has its constant yield value, and since the elongation change is positive in tension and negative in compression, we can write the increment of plastic work in the form

$$\Delta W_p = Y\,|\Delta e| \tag{37}$$

In particular, since bar 2 is the only one to yield in the present example, we may use the middle Eq. (2) to write

$$\Delta W_p = \begin{cases} Y|\Delta v| & \text{when bar 2 plastic} \\ 0 & \text{when bar 2 elastic} \end{cases} \tag{38}$$

Equation (38) shows that plastic work is done during each even-numbered stage and is always positive. The last column of Table 3a shows the cumulative plastic work through Stage 6. Clearly a further increment of $1.394\ Y^2/k$ must be added during each additional half cycle. Since the capacity of any real material to absorb plastic work is limited, the truss will become unserviceable after a relatively small number of cycles. The solid curve in Fig. 6 shows the accumulation of plastic work during the first few cycles. A loading program in which this phenomen occurs is called "alternating collapse".

In Sec. 2 we concluded that it was reasonable to neglect hardening at loads less than the yield-point load, at least for a single loading. To what extent is that conclusion valid in relation to shakedown? It turns out that the answer is quite different depending upon the type of hardening. We consider first an E/KH material. As before, bar 1 is always elastic, so we will be dealing with only two different incremental solutions:

$\Delta F_1 = k\Delta v/2$ always (39a)

$\Delta F_2 = k\Delta v$ $\Delta V = k\Delta v(1 + 1/\sqrt{2})$ bar 2 elastic (39b)

$\Delta F_2 = k'\Delta v$ $V = k\Delta v(r + 1/\sqrt{2})$ bar 2 plastic (39c)

During the initial loading (39b) will apply with $F_2 = Y$ to determine Stage 1L, then (39c) should be used with $\Delta V = (1.2 - 0.5\sqrt{2})Y$ for Stage 2L. Stage 3L is determined from (39b) with $\Delta F_2 = -2Y$, and Stage 4L from (39c) with $\Delta V = (-2.4 + \sqrt{2})Y$. Similar reasoning applies to Stages 5 and 6. The results are summarized in Table 3b. Since conditions are exactly the same at Stages 6L and 2L, further cycles will simply repeat the last four lines of Table 3b.

When a bar is plastic, the incremental work can be written

$$dW = F\,de = F\,\dot{e}\,dt = F\dot{F}/k'\,dt = d(F^2/2k') \tag{40}$$

However, to find the plastic work differential we must subtract the elastic differential $dW = d(F^2/2k)$. If F_0 and ΔF represent the beginning value and increment, respectively, of F during a plastic stage, we may write the plastic work increment in the form

$$\begin{aligned}\Delta W_p &= 0.5(1/k' - 1/k)[(F_0 + \Delta F)^2 - F_0^2] \\ &= (1 - r)(\Delta F/k')(F_0 + \Delta F/2)\end{aligned} \tag{41}$$

In particular, for bar 2 Eq. (41) becomes

$$\Delta W_p = (1 - r)\Delta v(F_0 + 0.5\Delta F) \tag{42}$$

The last two columns of Table 3b show the increments and cumulative values of the plastic work, and the dashed curve in Fig. 6 shows the cumulative plastic work over the first few cycles for $r = 0.1$.

Comparing the results for the E/PP and E/KH models we see that

the former is a reasonable approximation to the latt[illegible] parti- cular, both models predict that the plastic work wil[illegible] without bound as the cycles continue.

The E/IH material behaves quite differently. The increments are still governed by Eqs. (39), but the passive yield force comes from (19h) rather than from (19i). The resulting solution is exactly the same through Stage 3, but Stage 3L occurs when F_2 is equal to the negative of its value at Stage 2L, rather than when $\Delta F = -2Y$. The net result is that the truss spends more of its unloading time in the elastic Stage 3 and less in the partially-plastic Stage 4. The same thing happens during the reloading phase. Therefore, as shown in Table 3c, the solution at Stage 6L is different from that at Stage 2L, so that the next cycle of unloading-reloading will be different. In particular, we note that the value of F_2 at the end of each cycle has increased. Since this value is the current yield stress of bar 2, a larger proportion of each succeeding cycle is spent in a fully elastic state.

Results for the first few cycles are shown in Table 3c and by the dotted curve in Fig. 6. Although bar 2 is plastic during part of every cycle, it is apparent that the process is converging to an entirely elastic one, and that the total amount of plastic work is finite.

Let us return to the E/PP material and consider a different loading cycle. We begin in the same way by increasing V to 2.2Y, but we then superpose an alternating load at 45°. Therefore, Stages 1 and 2 are the same as in the previous example, but beginning with Stage 3 the load increments are defined by $\Delta H = -\Delta V = \Delta\lambda/\sqrt{2}$, where λ is cycled between 0 and 1.1Y. Therefore, Eqs. (1) can be written

$$\Delta F_1 - \Delta F_3 = \Delta\lambda \qquad \Delta F_1 + \Delta F_3 + \sqrt{2}\Delta F_2 = -\Delta\lambda \tag{43}$$

At the end of Stage 2 bar 2 is plastic, but the upward component of the diagonal load reduces F_2 so that Stage 3 is fully elastic. The increments of the three bar forces and two displacement components are all proportional to $\Delta\lambda$ and are given in the first column of Table 4. Although bars 1 and 3 are both in tension at the start of Stage 3, F_1 is increasing and F_3 is decreasing during the stage. Stage 3L is determined by bar 1 reaching its tensile yield force when $\lambda = (0.8 - 0.2\sqrt{2})Y = 0.517Y$.

In Stage 4 $\Delta F_1 = 0$ and Eqs. (7a) are valid for bars 2 and 3. Together with (43) this leads to $\Delta F_2 = 0$, also. The complete solution is shown in column 2 of Table 4. Stage 4 terminates when λ equals its final value of 1.1Y. The complete solution for Stage 4L is given in Table 5.

We now reduce the diagonal load so that $\Delta\lambda$ is negative. This causes a decrease in F_1, so that the truss is again fully elastic. As shown in Table 4, a negative $\Delta\lambda$ increases F_2 and bar 2 will yield in tension when $\lambda = 0.583$. The complete solution for the resulting Stage 5L is shown in Table 5. In stage 6 $\Delta F_2 = 0$ which leads to the same force increment values as in Stage 4 (see Table 4). However, bars 1 and 3 are now elastic and the corresponding Eqs. (7a) show that $\Delta u + \Delta v = 0$, whereas in Stage 4 $\Delta v = 0$. Thus, the displacement increments are different in the

two stages. Stage 6L is defined by $\lambda = 0$, and the resulting complete solution is given in Table 5.

The bar forces are all exactly the same at Stage 6L as they were at Stage 2L. Therefore, they will repeat the same sequence of values during each additional cycle of load. However, the vertical and horizontal displacements have each increased by $(0.3 + 0.2\sqrt{2})Y/k = 0.583Y/k$ during the cycle. Clearly they will increase by the same amount during each succeeding cycle. Therefore, although at any given instant the truss has at least two bars elastic and can support the given load, the deformations grow indefinitely with time and the truss eventually becomes unserviceable. This behavior has been termed "incremental collapse". A history of the displacements during the first few cycles is shown in Fig. 7.

Physically, we can observe that if bars 1 and 2 were both in plastic tension at the same time the truss would be a mechanism capable of a rotation about the end of bar 3 with $\dot{u} = \dot{v}$. Of course this does not occur, but in the course of a complete cycle bars 1 and 2 are plastic at different times. The net effect at the end of a complete cycle is a limited mechanism motion $\Delta u = \Delta v = 0.583\ Y/k$, as shown by the detailed analysis.

For this loading program the two stainhardening models are the same. Stage 2L is again given by the appropriate line from Table 3b or 3c. From then on the equilibrium equations are given by (43). Combining the kinematic and constitutive equations we obtain

$$E/ \quad \Delta F_1 = k(\Delta v + \Delta u)/2 \quad \Delta F_2 = k\Delta v \quad \Delta F_3 = k(\Delta v - \Delta u)/2 \tag{44a}$$

$$/SH \quad \Delta F_1 = k'(\Delta v + \Delta u)/2 \quad \Delta F_2 = k'\Delta v \quad \Delta F_3 = k'(\Delta v - \Delta u)/2 \tag{44b}$$

Stage 3 is fully elastic and terminates when bar 1 reaches its yield force, Y. The complete solution for this and subsequent stages are given in Table 6. In Stage 4 we use (44b) for bar 1 and (44a) for bars 2 and 3, and terminate the stage when λ reaches its maximum value, 1.1Y.

Decreasing λ in Stage 5 causes all bars to become elastic again. However, the stage does not end until bar 2 reaches its current yield stress, Y_2'. This value is the maximum of previous values of F_2, i.e., $Y_2' = 1.061$ which was attained at Stage 2L.

Stage 6 continues with bar 2 plastic and ends when λ is reduced to 0. Since Stages 6L and 2L do not have the same bar forces, if we want more stages we must find them explicitly. The qualitative description of each cycle is the same: as λ is increased all bars are elastic at the start but bar 1 becomes plastic before λ reaches 1.1Y; as λ is decreased all bars are first elastic but bar 2 becomes plastic before the cycle ends. However each time bar 1 or bar 2 yields its yield force is increased so that during the next cycle the truss will remain fully elastic until nearer the end of the cycle. Table 6 gives the complete results through Stage 14L, corresponding to three complete cycles.

Of particular interest are the total displacements added during each cycle. Using the results of Table 6 it is easy to show that the ratio of the increase in either u or v to its increase in the previous cycle is the constant value $\eta = 0.571636$.

Thus the net displacement can be written as a geometric series which converges to the final values

$$u = 1.508Y/k \qquad v = 2.932Y/k \tag{45}$$

Therefore, at least in this simple example, the introduction of strainhardening leads to finite values of the total displacements as compared with the infinite values predicted by the E/PP model.

5. REVERSE STRESSING UNDER MONOTONIC LOADING

In the previous sections we have seen various examples where the force ratios between bars will change as one or more bars becomes plastic, even though the external load is uniformly increasing. Here we present a simple example in which the ratios not only change magnitude, but actually change sign. The truss in Fig. 8a consists of three numbered vertical bars made of an E/PP material and joined to a perfectly rigid horizontal bar. The three vertical bars have equal areas A, lengths L and moduli E, but are assumed to have different yield stresses given by

$$Y_1 = Y \qquad Y_2 = 20Y \qquad Y_3 = 12Y \tag{46}$$

A vertical load P is applied as indicated. It is increased just to its yield-point value, and then decreased to zero. There are no horizontal loads, and we assume zero horizontal motion.

Static equations are obtained by considering vertical and moment equilibrium of the horizontal bar (Fig. 8b):

$$F_1 + F_2 + F_3 = P \qquad F_3 - F_1 = 0.5P \tag{47}$$

There are two degrees of kinematic freedom which we take to be the elongation v of bar 2 and the difference z between the elongations of bars 3 and 2, Fig. 8c. Clearly $z = L\theta$ where θ is the clockwise rotation of the horizontal bar. In terms of these variables the bar elongations are

$$e_1 = v - z \qquad e_2 = v \qquad e_3 = v + z \tag{48}$$

The constitutive behavior is given by the rate form of Eqs. (7a) and (7c). Combining these with Eqs. (48) we obtain

$$E/: \quad \dot{F}_1 = k(\dot{v} - \dot{z}) \qquad \dot{F}_2 = k\dot{v} \qquad \dot{F}_3 = k(\dot{v} + \dot{z}) \tag{49a}$$

$$/PP: \; F_i = \pm Y_i \qquad F_i\dot{e}_i \geq 0 \tag{49b}$$

where $k = AE/L$.

As P is first increased from 0 all three bars are elastic. Equations (47) and (49a) in integrated form lead easily to the solution

$$\text{Stage 1:} \quad F_1 = P/12 \qquad F_2 = 4P/12 \qquad F_3 = 7P/12$$
$$v = P/3k \qquad z = P/4k \tag{50}$$

Although bar 3 is the most highly stressed, bar 1 will be the first to yield since its yield force is so much smaller. Thus Stage 1 ends with the solution

$$\text{Stage 1L:} \quad F_1 = Y \qquad F_2 = 4Y \qquad F_3 = 7Y$$
$$v = 4Y/k \qquad z = 3Y/k \qquad P = 12Y \tag{51}$$

In Stage 2, F_1 maintains the known value Y, so that Eqs. (47) may be solved directly to obtain all bar forces. Bars 2 and 3 are elastic, so the last two Eqs. (49a) are available to

determine v and z. Further, since these bars have always been elastic, we may use the integrated form directly. Thus

$$\text{Stage 2:} \quad F_1 = Y \qquad F_2 = 0.5P - 2Y \qquad F_3 = 0.5P + Y$$
$$kv = 0.5P - 2Y \qquad kz = 3Y \tag{52}$$

This stage will end when bar 3 reaches its yield force 12Y:

$$\text{Stage 2L:} \quad F_1 = Y \qquad F_2 = 9Y \qquad F_3 = 12Y$$
$$v = 9Y/k \qquad z = 3Y/k \qquad P = 22Y \tag{53}$$

If bars 1 and 3 both remain plastic, the truss will be at its yield-point load. A formal solution of the equilibrium equations and the middle Eq. (49a) regains the values in (53) for the bar forces, load, and v, with z still undetermined. *However*, with $\dot{v} = 0$, the rate form of Eqs. (48) shows that $\dot{e}_1$ and $\dot{e}_3$ cannot both be positive, hence the inequalities in (49b) cannot both be *satisfied. The* same result is physically obvious from Fig. 8c. If bar 2 remains rigid, a yield mechanism would rotate the horizontal bar about the end of bar 2 which would result in a shortening of bar 1.

The above arguments show that in Stage 3 bar 3 will be plastic, but bar 1 will return to an elastic state. Thus we set $F_3 = 12Y$, solve (47) for the remaining forces, solve the first two (49a) for $\dot{v}$ and $\dot{z}$, and integrate using Eqs. (53) as initial conditions:

$$\text{Stage 3:} \quad F_1 = 12Y - 0.5P \qquad F_2 = 1.5P - 24Y \qquad F_3 = 12Y$$
$$\dot{v} = 1.5\dot{P}/k \qquad \dot{z} = 2\dot{P}/k \tag{54}$$
$$kv = 1.5P - 24Y \qquad kz = 2P - 41Y$$

The force in bar 1 is now decreasing, and it will reach its yield value in compression while bar 2 is still elastic:

$$\text{Stage 3L:} \quad F_1 = -Y \quad F_2 = 15Y \quad F_3 = 12Y \quad P = 26Y \tag{55a}$$

$$kv/Y = 15 \qquad kz/Y = 11 \tag{55b}$$

If the load is maintained at 26Y with bars 1 and 3 remaining plastic, the solution consists of Eqs. (55a), $kv/Y = 15$, and z is undetermined with any $z > 0$ satisfying both relevant inequalities in (49b). Therefore Eqs. (55) represent the beginning of the yield-point solution.

Instead of maintaining P at 26Y, let us immediately reduce it. In most problems a reversal of the only load will cause all plastic bars to become elastic. However, if we superimpose the elastic solution (50) with a negative load increment on Eqs. (55), the force in bar 1 will immediately exceed its compressive yield strength. Therefore, even though bar 1 has just reached yield under an increasing load, it will remain at yield when we decrease the load. The resulting solution is easily found to be

$$\begin{aligned} \text{Stage 4:} \quad & F_1 = -Y \qquad F_2 = 0.5P + 2Y \quad F_3 = 0.5P - Y \\ & kv = 2Y + 0.5P \quad kz = 11Y \end{aligned} \tag{56}$$

Clearly no further yielding takes place as P is reduced to zero. When the truss is fully unloaded it has a set of residual forces and permanent displacements given by

$$\text{Stage 4L:} \quad P = 0 \quad F_1 = F_3 = -Y \quad F_2 = 2Y \quad kv/Y = 2 \quad kz/Y = 11 \tag{57}$$

The behavior of this seemingly simple structure becomes even more complex when a strainhardening material is considered. Under an increasing load the elastic behavior and Stage 1L will, of course, be the same. In Stage 2 bar 1 will be in plastic tension, and the stage will end when bar 3 reaches its yield force of 12Y. Bar 1 will unload and bar 3 will be in plastic tension in Stage 3. Thus far the solution will be only slightly different from the perfectly plastic truss because of the effects of strainhardening when bars 1 or 3 are plastic.

The termination of Stage 3 will depend upon the type of hardening. For kinematic hardening bar 1 will yield in compression when the change in F_1 from its value at Stage 2L is -2Y which occurs at a load P = 28.9Y. On the other hand, for an isotropic hardening material bar 1 will yield when its force reaches the negative of its previous maximum value at P = 31.2.

Figure 9 shows the relation between the load P and the deflection

$$\delta = v - z/2 \tag{58}$$

of the point of load application. The unloading portion shows unloading from Stage 3L. Part of the difference between the curves is caused by the fact that they show unloading from different values of the load.

In the presence of stainhardening there is no maximum allowable load. Therefore, instead of reducing the load from Stage 3L, let us form Stage 4 by continuing to increase it. Bar 1 will be in plastic compression, bar 3 in plastic tension, and bar 2 will be still elastic with increasing force. The stage will end when F_2 reaches its maximum value of 20Y. Table 7 shows the resulting solutions for both types of hardening at the various limit stages, assuming r = 0.1.

In Stage 5, bar 2 will be plastic. If all bars were plastic, the bar force increments would be in the same proportion as in Stage 1 when all bars were elastic. In particular, ΔF_1 would be positive which would mean that it would no longer be in plastic compression. Therefore, in Stage 5 bar 1 will again be elastic, while bars 2 and 3 are in plastic tension. Stage 5L occurs when bar 1 reaches its current tensile yield stress. For the KH material this will be when ΔF_1 = 2Y, whereas for an IH material it is not until F_1 reaches the same magnitude tensile force it had in compression at Stage 4L. As shown in Table 7 and Figure 10, the load and displacement at Stage 5L are quite different for the different types of hardening.

However, this difference is really qualitative but not quantitative. To see this, let us continue to increase the load to a value of 80Y. During this Stage 6 all bars are plastic in tension. Table 7 shows that at the final load all bar forces and displacements have only very small differences, and the two curves are virtually indistinguishable in Fig. 10.

Let us review the behavior of bar 1 of this simple strain-hardening truss. As the single load P is monotonically increased, bar 1 is first elastic, then yields in plastic tension, then unloads and yields in plastic compression, then reloads, and finally yields for a second time in plastic tension.

To conclude this section we consider one more loading program in which P is increased until bar 2 just reaches yield and is than decreased to zero. Stage 4L is the same as before, but a new Stage 5* will result from the unloading. Since fully elastic unloading would decrease F_1 which is already at compressive yield, Stage 5* will find bar 1 plastic and bars 2 and 3 elastic. But the original Stage 5 also had bar 1 in plastic compression. In other words, once the truss is at Stage 4L, bar 1 will continue to yield in plastic compression whether the load is increased or decreased!

6. NON-UNIQUENESS

We return to the truss of Fig. 1, with the following modification: the bars have unequal yield forces with $Y_1 = Y_3 = Y$, $Y_2 = 3Y$. We consider a load-controlled program with H = 0 and V increasing to the yield-point load. However, we do not make any assumptions of symmetry. The bars are all E/PP.

For convenience we shall repeat the defining equations in a form specialized to the particular loading program. The equilibrium equations are

$$F_1 + F_3 + \sqrt{2}F_2 = \sqrt{2}V \qquad (59a)$$

$$F_1 - F_3 = 0 \qquad (59b)$$

It turns out that bar 2 is always elastic and bars 1 and 3 are either elastic or in plastic tension. Therefore, we may combine the kinematic and constitutive equations to obtain

$$F_2 = kv \tag{59c}$$

$$\text{EITHER } F_1 = (k/2)(v + u) \quad \text{OR} \quad F_1 = Y \tag{59d}$$

$$\text{EITHER } F_3 = (k/2)(v - u) \quad \text{OR} \quad F_3 = Y \tag{59e}$$

together with inequality conditions which may be written

$$F_1 \leq Y \qquad F_2 \leq Y \qquad |\dot{u}| \leq \dot{v} \tag{59f}$$

Equations (59) provide a total of five equations to determine F_1, F_2, F_3, u, and v.

Stage 1, of course, is elastic, so we use the first branch in Eqs. (59d, e) to obtain the unique solution

$$u = 0 \qquad 2F_1 = 2F_3 = F_2 = kv = (2 - \sqrt{2})V \tag{60}$$

as in Sec. 2. However, with the stronger bar 2, bars 1 and 3 yield first, so that Stage 1L is

$$u = 0 \qquad F_1 = F_3 = Y \qquad F_2 = kv = 2Y \qquad V = (2 + \sqrt{2})Y \tag{61}$$

In Stage 2 bars 1 and 3 are both plastic so the second branch is used in (59d, e). However, this means that (59b) is satisfied identically. We can still use (59a, c) to obtain the unique values

$$F_1 = F_3 = Y \qquad F_2 = kv = V - \sqrt{2}Y \tag{62a}$$

but we cannot determine the value of the horizontal displacement u, although (59f) does provide the bounds

$$|u| < \Delta V/k \tag{62b}$$

In particular, Stage 2L at the yield-point load is

$$F_1 = F_3 = Y \qquad F_2 = kv = 3Y \qquad V = (3 + \sqrt{2})Y \tag{63a}$$

$$|u| \leq Y/k \tag{63b}$$

Figure 11 shows the unique solution at Stage 1L and several solutions for Stage 2L. For example, Fig. 11b shows the largest allowable value of u, u = Y/k. In this solution bar 3 rotates as a rigid body, bar 2 elongates elastically, and bar 1 elongates plastically. Notice that any value of u larger than Y/k would require a shortening of bar 3 which is not permissible when it is yielding in tension. On the other hand any position of point D between the limiting ones in Figs. 11 a and b requires a lengthening of both bars 1 and 3 which is consistent with their both being at tensile yield.

Mathematically, the non-uniqueness was caused by the fact that two bars yielded simultaneously and reduced one of the equilibrium equations to an identity. In reality, of course, infinitesimal differences between the two bars would make it extremely unlikely that they would both yield at exactly the same instant. To examine this facet of the problem, let us make a perturbation of the problem by taking $Y_3 = Y + X$ where X is positive but otherwise arbitrary. Stages 1 and 1L will be the same as before, although only bar 1 will yield at Stage 1L. In Stage 2 we now use the first branch of (59e) along with the second branch of (59d). The resulting unique solution consists of (62a) together with the value

$$u = \Delta V/k \tag{64}$$

In particular, at Stage 2L the solution consists of Eqs. (63a) plus

$$u = Y/k \tag{65}$$

Figure 11b shows the resulting configuration. This solution is completely independent of X, provided only that X is positive. In particular, X may be arbitrarily small so that there is no practical difference between the two yield forces.

However, it is clear that if we leave $Y_3 = Y$ and set $Y_1 = Y + X$, the solution at Stage 2L will consist of Eqs. (63a) plus

$$u = -Y/k \tag{66}$$

as pictured in Fig. 11a. This solution, too, is valid for arbitrarily small X. Therefore, although the non-unique solution of the original problem can be made unique by an arbitrarily small perturbation, two different perturbations give two very different unique solutions.

Let us now consider the effect of strainhardening. In doing so we shall allow for different rates of hardening in bars 1 and 3 by defining

$$k_1' = kr_1 \qquad k_3' = kr_3 \tag{67}$$

Then Eqs. (59d, e) will be replaced by

$$\text{IF } F_1 \le Y \text{ THEN } F_1 = k(u + v)/2 \text{ ELSE } F_1 = r_1k(u + v)/2 \tag{68a}$$

$$\text{IF } F_3 \le Y \text{ THEN } F_3 = k(v - u)/2 \text{ ELSE } F_3 = r_3k(v - u)/2 \tag{68b}$$

Stages 1 and 1L are the same as before, and the complete solution for Stages 2 and 2L is easily obtained. Unique values are obtained for all variables, and if the strainhardening tends to zero the bar forces and vertical displacement will tend to the values in Eqs. (62a) and (63a). We shall focus our attention on the unique value of u at Stage 2L which is given by

$$u = \frac{r_1 - r_3}{r_1 + r_3}\frac{Y}{k} \tag{69}$$

If the strainhardening in the two bars is the same, $r_1 = r_3$, we obtain the symmetric solution u = 0 as pictured in Fig. 11c. On the other hand, if $r_1 = 0$, and only bar 3 hardens, we get $u = -Y/k$ as in Eq. (66) and Fig. 11a. Likewise, if only bar 1 hardens the solution is again Eq. (65), Fig. 11b. Clearly unequal non-zero hardening in the two bars can produce any value between these two extremes; for example $r_1 = 0.1$, $r_3 = 0.05$ gives the result $u = Y/3k$ as shown in Fig. 11d.

7. CONCLUSIONS

In the preceding sections we have used two very simple trusses to illustrate several important concepts in plasticity. The ideas presented are not new, of course, nor are most of the applications. The truss in Fig. 1 has been used many times starting at least as early as 1948 [1, 2, 3]* to illustrate plastic and other inelastic behavior. It has also been used to illustrate some singular features in plastic design [4]. The truss in Sec. 5 (Fig. 8) was first introduced by Drucker [5], and much of the present development was taken from that reference.

Countless texts on plasticity are available [6, 7, 8, 9, 10] (to name just a few). The basic constitutive equations presented in Secs. 1 and 2 can be generalized to two and three dimensions and to various structural problems. For the particular

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* Numbers in square brackets refer to references collected at the end of the paper.

case of a simply-supported circular plate generalizations of the four stress-strain curves shown in Fig. 2 have been applied [11]. Figure 12, taken from Ref. [11] shows the relation between the pressure and the displacement of the plate center. The results are certainly qualitatively similar to those for the three-bar truss as shown in Fig. 3.

Shakedown was first introduced by Melan [12]. Many results have been presented by Symonds and by Neal [13, 14, 10]. Important theoretical work has been done by Koiter [15, 16]. Recently, an entire book by Gokhfeld and Cherniavsky [17] has been devoted to the subject.

The idea that plastic stress reversal can occur even under *a single monotonic loading has many important implications. For* example, the somewhat simpler theories known as "plastic-deformation" theories which directly relate stress to strain rather than relating their rates is obviously inappropriate when this phenomenon occurs.

Figure 13 shows an eleven-bar truss which does not look particularly unusual. The top four bars all have the same cross-section and are made of an E/KH material with $r = 0.1$. However their yield forces are

$$Y_1 = Y \qquad Y_2 = Y_3 = 12Y \qquad Y_4 = 5Y \tag{70}$$

The bottom bars are all substantially stiffer and have a yield stress high enough to prevent yielding. As the load P is increased, bar 1 first yields in tension, then bar 4 reaches its yield force 5Y. Bar 1 must now unload, and it then yields in compression. Next, bar 2 yields in tension, and bar 1 again

unloads, goes into tension, and eventually yields again in tension. Table 8 shows the bar forces, load, and displacement of the point of load application for the following specific values:

$$\begin{aligned} &\text{Bars 1-4:} \quad A = 100\ \text{mm}^2 \quad E = 80\ \text{GPa} \quad E' = 8\ \text{GPa} \\ &\text{Bars 5-11:} \quad A = 1000\ \text{mm}^2 \quad E = 240\ \text{GPa} \\ &Y = 10\ \text{MPa} \end{aligned} \tag{71}$$

A similar stress reversal has been observed in elastic-plastic torsion of some hollow bars [18, 19]. Shaw [20] has numerically solved the torsion of a bar whose cross section is a hollow rectangle with fillets. He has shown that plastic behavior starts on the inner boundary at the fillet, and that the stress vector *there is in the direction of the torque. However, it was first* pointed out in [18] and later numerically verified in [19] that at the limiting plastic torque the stress vector there must be in the opposite direction from the torque. An unfortunate result of this plastic stress reversal is that the Nadai sandhill-soapfilm analogy [21] is no longer applicable.

The phenomenon of non-uniqueness was encountered, apparently for the first time, in the development of a finite-element model with discontinuous displacements for use in plane-strain plasticity [22, 23]. Since this was a primarily numerical development, it was not clear whether the non-uniqueness was inherent in the E/PP model, or whether it was a peculiarity of the particular finite-element model. The analysis presented in Sec. 6 [24, 25] shows that it is, indeed, a possibility which must be considered in any problem using the E/PP model.

Figures 14 and 15 [24] show some other simple examples. The elastic solution for the truss in Fig. 14 is symmetric. It ends when the two side bars reach yield. Further increase of load can be associated with any non-negative change in the lengths of the vertical bars. Figure 14 shows the two extreme positions where one of the plastic bars does not change its length. Any intermediate solution can be obtained by a rigid-body rotation of the upper triangle about the center-point of the truss.

As the load on the frame in Fig. 15 is increased, hinges will form first at C, then at B, and then simultaneously at points A and D, but the frame will not collapse until the final hinges form at E. The solution is unique during the elastic and first two partly-plastic stages, but once the hinges form at A and D, the rotations at these new hinges are controlled only by inequalities, both being required to exhibit tensile strain at the top of the hinge. Figure 15 shows the two extreme positions. Any intermediate solution obtained by a rigid-body vertical motion of the central part of the frame is also possible.

The possibility of part of the solution being non-unique has important implications in finite-element programs. Even if the engineer is concerned only with forces which are unique, a non-unique solution corresponds to a singular set of equations, and hence to a singular stiffness matrix. Thus it may be impossible to continue the solution past the point where the non-uniqueness occurs, even though the load is still below the yield-point load. Numerical round-off error may mask the

singularity of the stiffness matrix, but the resulting problem would be highly ill-conditioned and might lead to misleading results. This problem is certainly worthy of further investigation.

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Stage	kv/Y	F_1/Y	F_2/Y
1L	1	1/2	1
2L	2	1	$1 + r$
3L	8	$1 + 3r$	$1 + 7r$
4L	$6 - 14r$	$-4r$	$-(1 + 7r)$
5L	$4 - 12r$	$-(1 + 3r)$	$-(1 + 9r - 2r^2)$
6L	-6	$-(1 + 8r - 6r^2)$	$-(1 + 19r - 14r^2)$
7L	$-4 + 38r - 28r^2$	$11r - 8r^2$	$1 + 19r - 14r^2$
8L	0	$2 - 8r + 6r^2$	$1 + 23r - 52r^2 + 28r^3$

Table 1

Three-bar truss with isotropic hardening

Stage	kv/Y	F_1/Y	F_2/Y
1L	1	1/2	1
2L	2	1	1
3L	8	1	1
4L	6	0	-1
5L	4	-1	-1
6L	-6	-1	-1
7L	-4	0	1
8L	-2	1	1
9L	0	1	1

Table 2

Three-bar perfectly-plastic truss

Cycle	Stage	V/Y	kv/Y	F_2/Y	$\Delta W_p k/Y^2$	$\Sigma W_p k/Y^2$
0	1L	1.707	1.000	1.000	0	0
	2L	2.200	1.697	1.000	0.697	0.697
1	3L	-1.214	-0.303	-1.000	0	0.697
	4L	-2.200	-1.697	-1.000	1.394	2.091
	5L	1.214	0.303	1.000	0	2.091
	6L	2.200	1.697	1.000	1.394	3.485

Table 3a

Alternating Load: E/PP

Cycle	Stage	V/Y	kv/Y	F_2/Y	$\Delta W_p k/Y^2$	$\Sigma W_p k/Y^2$
0	1L	1.707	1.000	1.000	0	0
	2L	2.200	1.611	1.061	0.566	0.566
1	3L	-1.214	-0.389	-0.939	0	0.566
	4L	-2.200	-1.611	-1.061	1.099	1.666
	5L	1.214	0.389	0.939	0	1.666
	6L	2.200	1.611	1.061	1.099	2.765

Table 3b

Alternating Load: E/KH

Cycle	Stage	V/y	kv/Y	F_2/Y	$\Sigma W_p/kY^2$
0	1L	1.707	1.000	1.000	0
	2L	2.200	1.611	1.061	0.566
1	3L	-1.423	-0.511	-1.061	0.566
	4L	-2.200	-1.474	-1.157	1.528
	5L	1.752	0.840	1.157	1.528
	6L	2.200	1.396	1.213	2.121
2	7L	-1.941	-1.030	-1.213	2.121
	8L	-2.200	-1.351	-1.245	2.475
	9L	2.051	1.139	1.245	2.475
	10L	2.200	1.324	1.263	2.684
3	12L	-2.200	-1.309	-1.274	2.806
	14L	2.200	1.301	1.280	2.877
4	16L	-2.200	-1.296	-1.284	2.918
	18L	2.200	1.293	1.286	2.941
5	20L	-2.200	-1.291	-1.287	2.955
	22L	2.200	1.290	1.288	2.963
6	24L	-2.200	-1.289	-1.288	2.967
	26L	2.200	1.289	1.288	2.970

Table 3c

Alternating Load: E/IH

	All bars elastic	bar 1 plastic	bar 2 plastic
$\Delta F_1/\Delta\lambda$	$1 - 1/\sqrt{2}$	0	0
$\Delta F_2/\Delta\lambda$	$1 - \sqrt{2}$	0	0
$\Delta F_3/\Delta\lambda$	$-1/\sqrt{2}$	-1	-1
$k\Delta u/\Delta\lambda$	1	2	1
$k\Delta v/\Delta\lambda$	$1 - \sqrt{2}$	0	-1

Table 4

Summary of incremental solutions
for diagonal load

Cycle	Stage	λ/Y	F_1/Y	F_2/Y	F_3/Y	ku/Y	kv/Y
0	1L	-	0.500	1.000	0.500	0	1.000
	2L	0	0.849	1.000	0.849	0	1.697
1	3L	0.517	1.000	0.786	0.483	0.517	1.483
	4L	1.100	1.000	0.786	-0.100	1.683	1.483
	5L	0.583	0.849	1.000	0.266	1.116	1.697
	6L	0	0.849	1.000	0.849	0.583	2.280
one	6L-2L	0	0	0	0	0.583	0.583

Table 5

Diagonal Load: E/PP

Cycle	Stage	λ/Y	F_1/Y	F_2/Y	F_3/Y	ku/Y	kv/Y
0	1L	-	0.500	1.000	0.500	0	1.000
	2L	0	0.805	1.061	0.804	0	1.611
1	3L	0.470	1.000	0.786	0.335	0.665	1.335
	4L	1.100	1.072	0.684	-0.483	2.202	1.234
	5L	0.457	0.805	1.061	0.159	1.292	1.611
	6L	0	0.765	1.118	0.765	0.646	2.177
2	7L	0.740	1.072	0.684	0.025	1.692	1.743
	8L	1.100	1.113	0.626	-0.443	2.571	1.685
	9L	0.261	0.765	1.118	0.396	1.385	2.177
	10L	0	0.742	1.150	0.742	1.015	2.500
3	11L	0.894	1.113	0.626	-0.152	2.280	1.976
	12L	1.100	1.136	0.593	-0.419	2.782	1.943
	13L	0.149	0.742	1.150	0.531	1.437	2.500
	14L	0	0.729	1.169	0.729	1.226	2.685

Table 6

Diagonal Load: E/SH

Stage	1L	2L	3L		4L		5L		6L		5L*	
Bar States*	EEE	TEE	EET		CET		ETT		TTT		CEE	
Hardening			K	I	K	I	K	I	K	I	K	I
F_1/Y	1	1.3	-0.7	-1.3	-1.5	-1.7	0.5	1.7	2.6	2.7	-2.7	-2.8
F_2/Y	4	8.1	15.8	18.2	20.0	20.0	26.2	30.3	34.8	34.6	5.3	5.6
F_3/Y	7	12.0	13.8	14.3	15.4	15.0	27.6	35.3	42.6	42.7	-2.7	-2.8
kv/Y	4	8.1	15.8	18.1	20.0	20.0	103.3	123.3	167.7	165.5	5.3	5.6
ku/Y	3	3.9	13.7	16.7	26.2	22.0	99.9	122.0	150.5	153.7	22.9	18.7
P/Y	12	21.4	28.9	31.2	33.9	33.3	54.3	67.3	80.0	80.0	0	0
kδ/Y	5.5	12.6	22.7	26.6	34.1	31.0	125.1	184.3	242.9	242.4	16.7	14.9

Table 7

Three-bar truss of Figure 8

* E = Elastic, T = Plastic tension, C = Plastic Compression

Stage	1L	2L	3L	4L	5L
Status	all E	1T	4T	4T,1C	4T,2T
F_1 (kN)	1.00	1.09	-0.91	-1.17	0.83
F_2 "	1.96	2.46	10.59	12.00	16.52
F_3 "	0.92	1.23	7.73	8.82	11.07
F_4 "	4.31	5.00	8.19	8.96	18.15
F_5 "	-1.00	-1.09	0.91	1.17	-0.83
F_6 "	-4.31	-5.00	-8.19	-8.96	-18.15
F_7 "	1.41	1.54	-1.29	-1.66	1.17
F_8 "	-1.41	-1.54	1.29	1.66	-1.17
F_9 "	4.30	5.23	17.03	19.16	28.76
F_{10} "	6.09	7.07	11.59	12.67	25.67
F_{11} "	2.00	2.17	-1.83	-2.34	1.66
P (kN)	0.73	0.87	2.02	2.25	3.85
u (mm)	0.06	0.12	-1.62	-2.18	2.93
v (mm)	0.96	1.16	7.06	8.34	25.89

Table 8

Solution for truss of Fig. 13

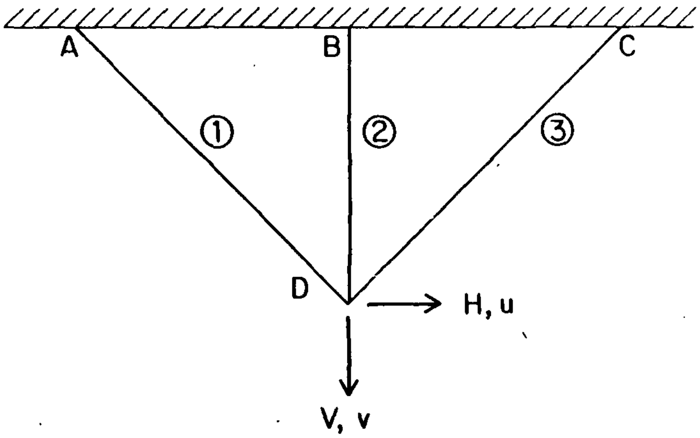

Fig. 1 Three-bar truss.

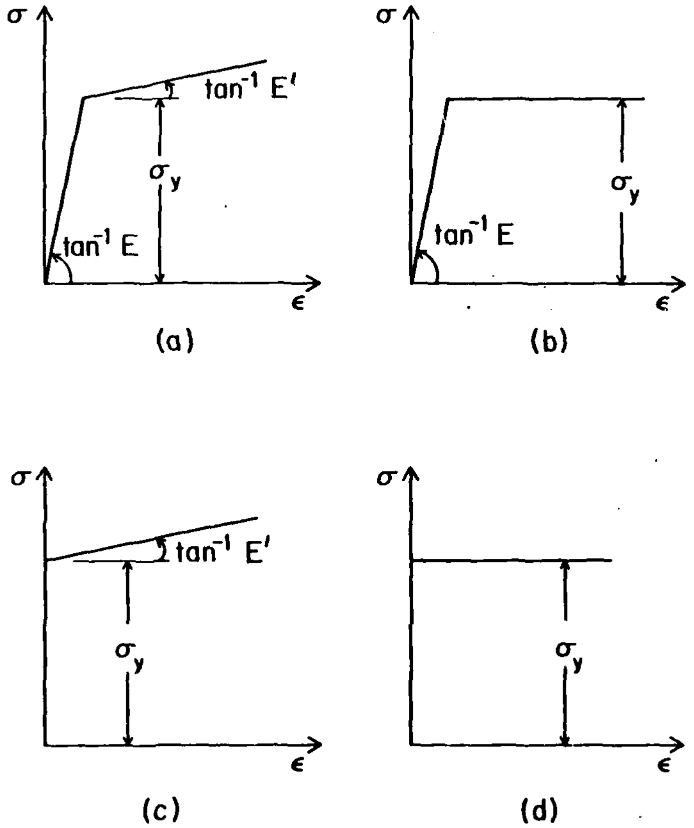

Fig. 2 Stress-strain curves.
(a) Elastic/strain-hardening (E/SH)
(b) Elastic/perfectly-plastic (E/PP)
(c) Rigid/strain-hardening (R/SH)
(d) Rigid/perfectly-plastic (R/PP)

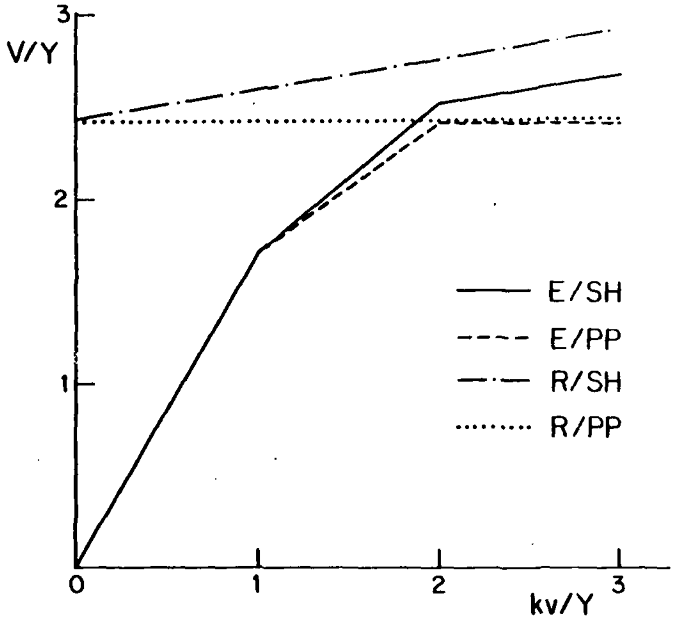

Fig. 3 Load-deformation curves.

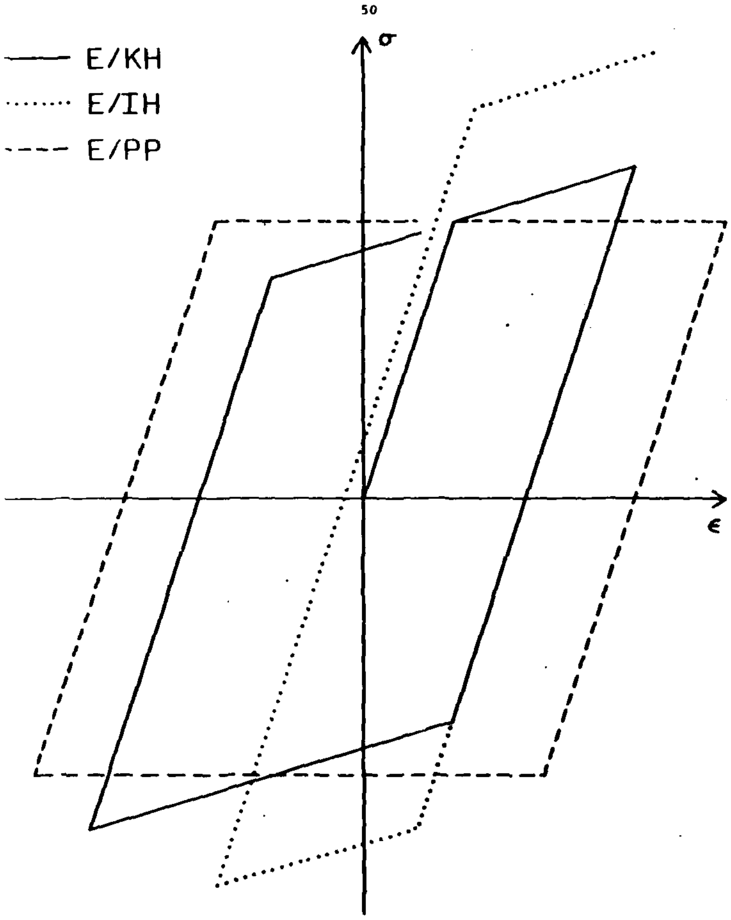

Fig. 4 Stress-strain curves for unloading and reloading.

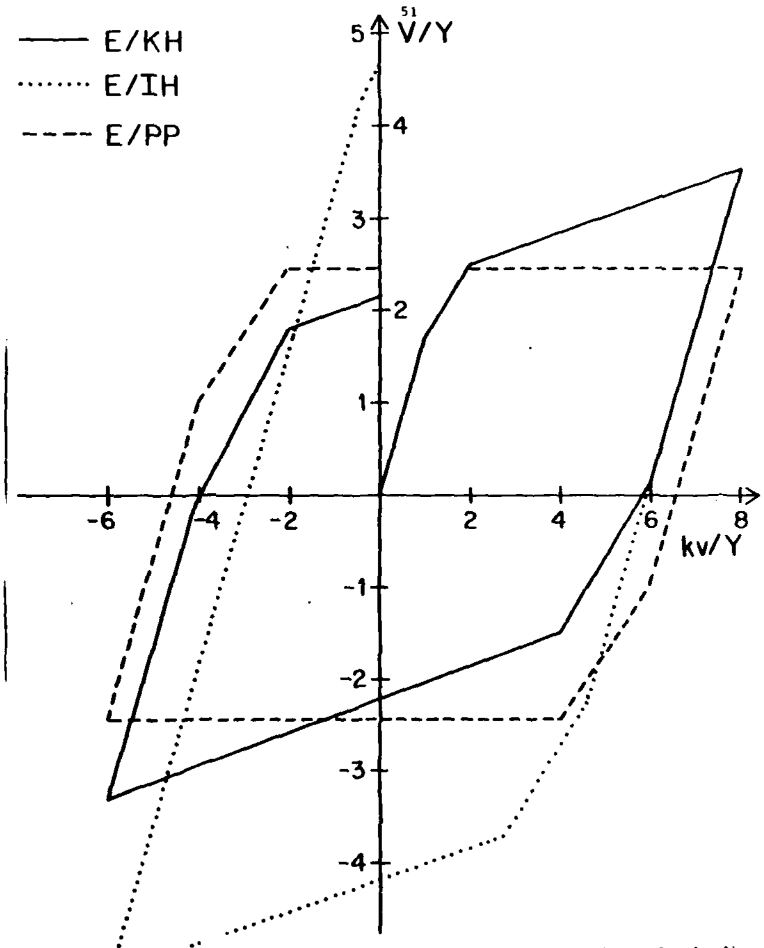

Fig. 5 Load-deformation curves for unloading and reloading.

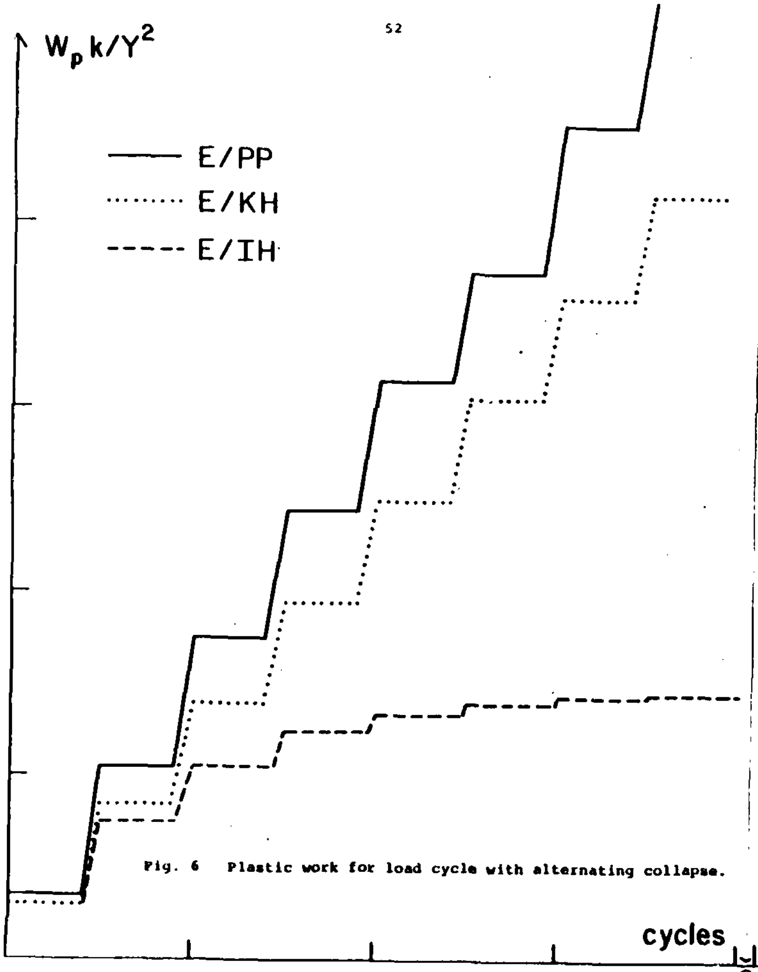

Fig. 6 Plastic work for load cycle with alternating collapse.

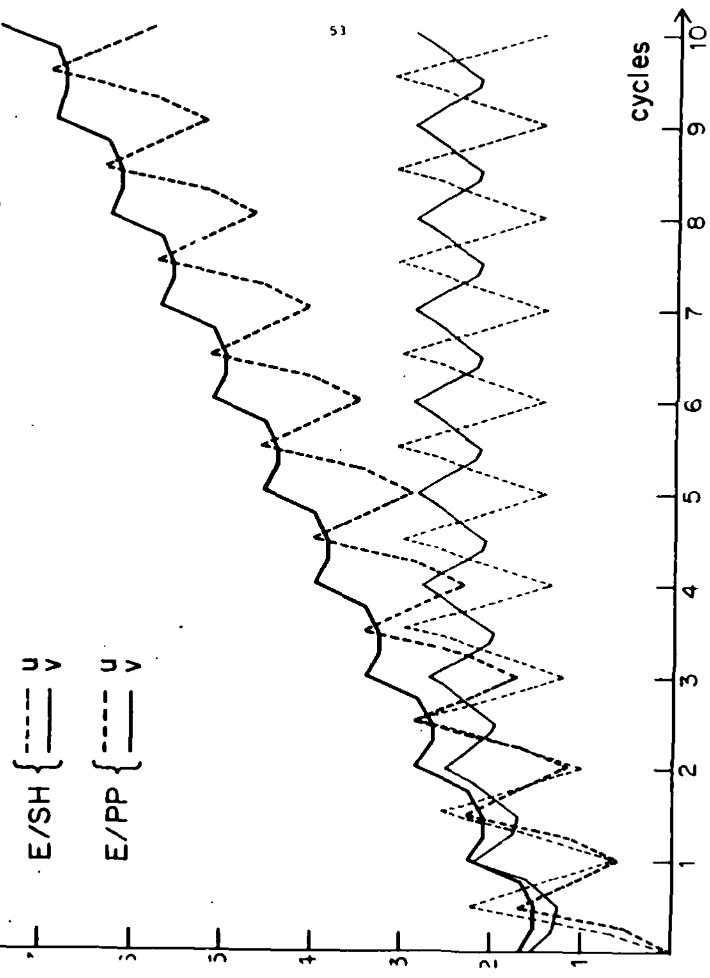

Fig. 7 Deformations for load cycle with incremental collapse.

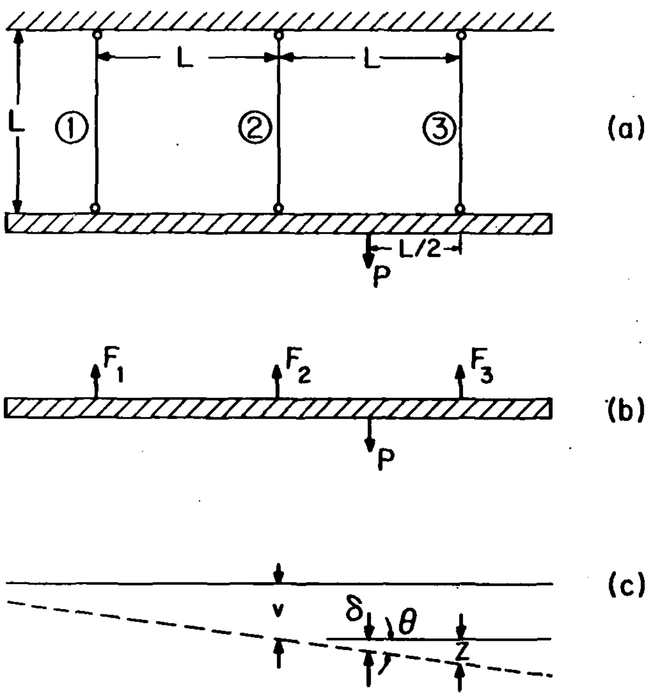

Fig. 8 Truss with three vertical bars.
(a) Loaded truss
(b) Statics
(c) Kinematics

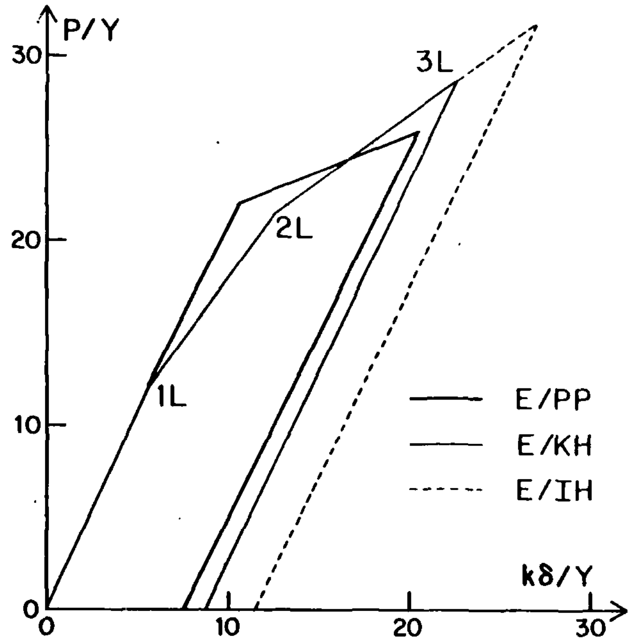

Fig. 9 Load-deflection curves.

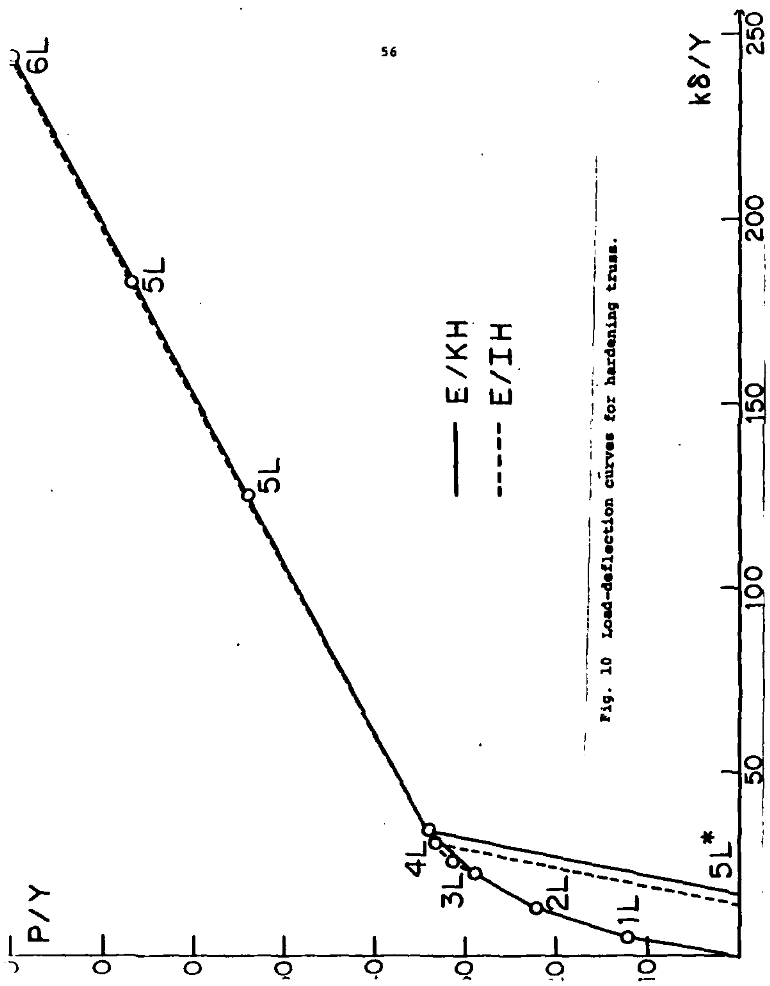

Fig. 10 Load-deflection curves for hardening truss.

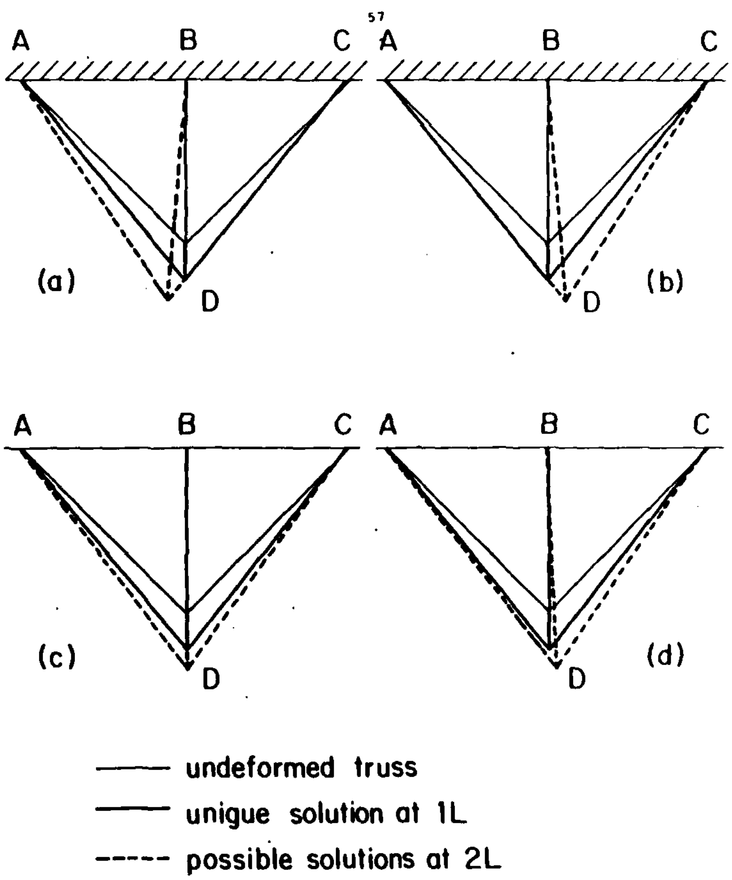

Fig. 11 Non-unique truss configurations.
(a) minimum horizontal displacement
(b) maximum horizontal displacement
(c) symmetric solution
(d) generic solution

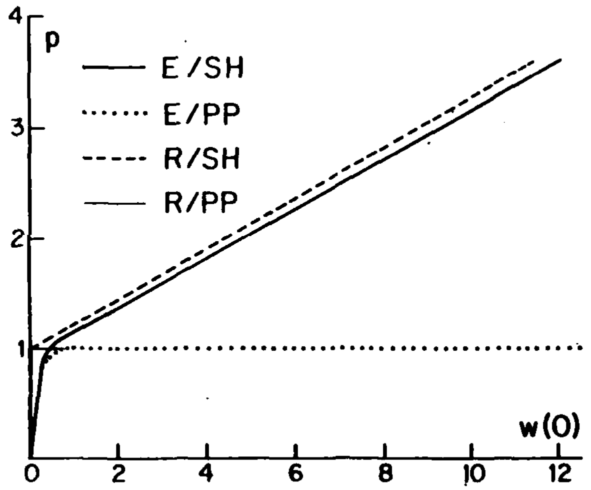

Fig. 12 Load-deformation curves for circular plate.

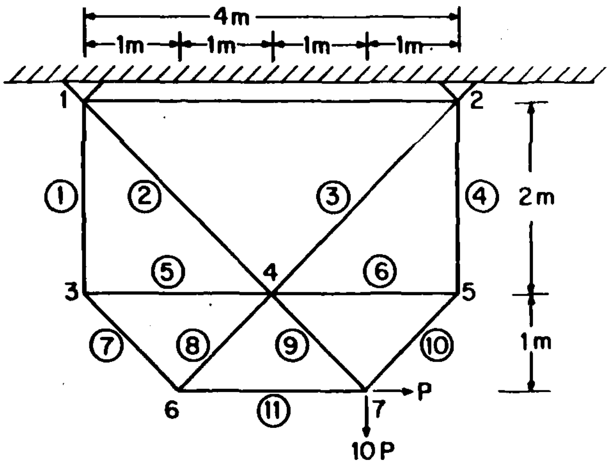

Fig. 13 Eleven-bar truss.

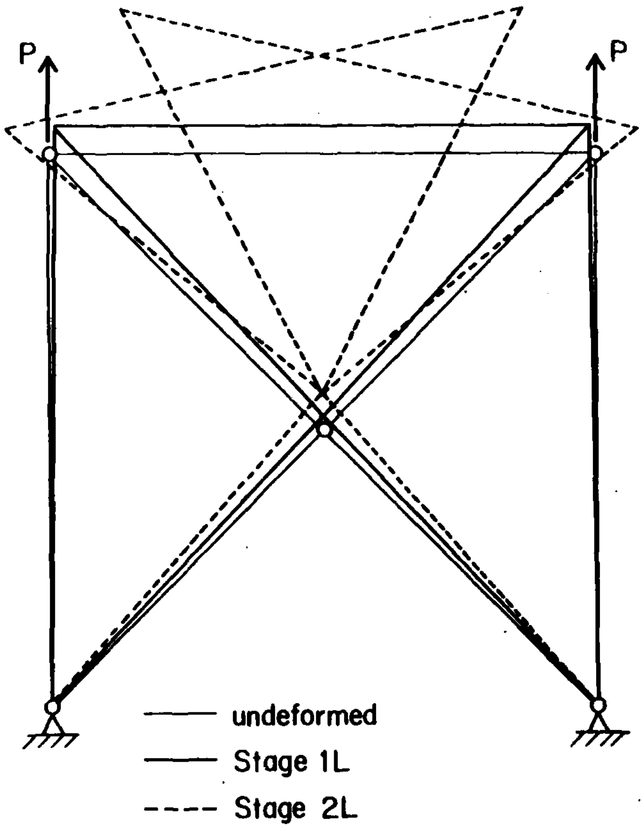

Fig. 14. Seven-bar truss.

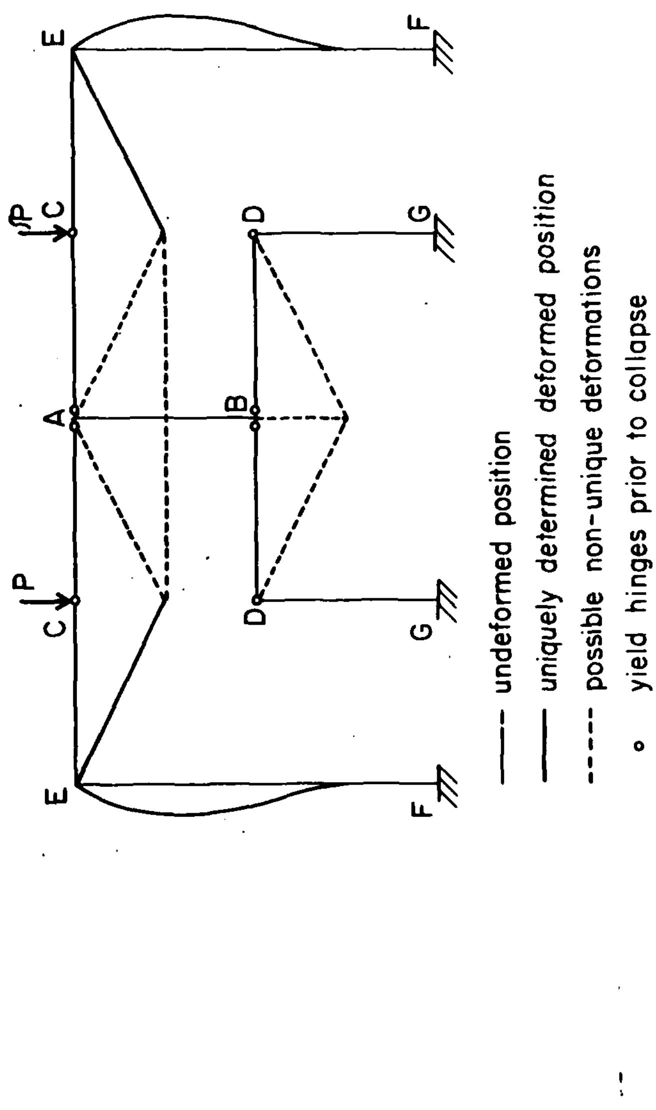

Fig. 15 Plane frame.

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